Elasto-Plastic Stresses in Thick Walled Cylinders

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In the optimal design of a modern gun barrel, there are two main objectives to be achieved: increasing its strength-weight ratio and extending its fatigue life. This can be carried out by generating a residual stress field in the barrel wall, a process known as autofrettage. It is often necessary to machine the autofrettaged cylinder to its final configuration, an operation that will remove some of the desired residual stresses. In order to achieve a residual stress distribution which is as close as possible to the practical one, the following assumptions have been made in the present research on barrel analysis: A von Mises yield criterion, isotropic strain hardening in the plastic region in conjunction with the Prandtl-Reuss theory, pressure release taking into consideration the Bauschinger effect and plane stress conditions. The stresses are calculated incrementally by using the finite difference method, whereby the cylinder wall is divided into N-rings at a distance Δr apart. Machining is simulated by removing rings from both sides of the cylindrical surfaces bringing the cylinder to its final shape. After a theoretical development of the procedure and writing a suitable computer program, calculations were performed and a good correlation with the experimental results was found. The numerical results were also compared with other analytical and experimental solutions and a very good correlation in shape and magnitude has been obtained. [DOI: 10.1115/1.1593078]

Introduction

Increased strength-weight ratio and extended fatigue life are the main objectives of optimal design for the modern gun barrel. These can be carried out by generating a residual stress field in the barrel wall, a process known as autofrettage.

There are two principal autofrettage processes. One is "hydrostatic," based on the use of high hydrostatic pressure that is applied inside the tube and the second is "swaging," which is based on the use of an oversized mandrel forced through the tube bore, thereby causing it to expand.

The residual stresses, which have a compressive effect on the internal barrel surface, increase the barrel elastic strength pressure and slow down the development of cracks. Furthermore, it is sometimes necessary to machine the barrel to its final dimensions, which reduces the contribution of residual stresses.

Since the basic data for calculating elastic strength pressure and fatigue life are derived from a precise distribution of the residual stresses, a theoretical solution for these stresses must be as close as possible to the practical one. Although there are many solutions which deal with the problem of determining stress distribution, none of them are completely satisfactory, since they are all based on certain assumptions on yield criteria and other effects.

The simplest and most general theoretical treatment of the partially plastic thick-walled cylinder has been the use of the Tresca yield criterion, and the assumption of the elastic-perfectly plastic material, without the Bauschinger effect [1–3].

Using the von Mises or Tresca yield criteria together with a certain Bauschinger effect parameter [4–8], a more precise solution for residual stress was developed, and a good correlation with the experimental results was achieved.

The basic assumptions of the present research for the barrel analysis are: the von Mises yield criterion, isotropic strain hardening in the plastic region in conjunction with the Prandtl-Reuss theory, pressure release taking into account the Bauschinger effect [9], and plane stress conditions (often called "open end" conditions).

It is also assumed that the longitudinal stresses in the swage autofrettage caused by frictional forces would not affect the results of a two-dimensional calculation compatible with hydrostatic autofrettage.

After the theoretical development of the procedure and the subsequent writing of an appropriate computer program that includes a machining simulation, calculations were performed with results exhibiting a good correlation with the experimental data.

Analytical Solution

As it was previously assumed, the autofrettage is performed when the cylinder remains open, namely plane stress conditions prevail $(\sigma_z=0)$. Under this condition the elastic stresses are given by the Lamé equations [10],

$$\sigma_\theta=\frac{a^2P}{b^2-a^2}\left(1+\frac{b^2}{r^2}\right);\quad \sigma_r=\frac{a^2P}{b^2-a^2}\left(1-\frac{b^2}{r^2}\right). \tag{1}$$

In the inelastic region, the total strain is composed of elastic and plastic strains,

$$\varepsilon_r=\varepsilon_r^e+\varepsilon_r^p,\quad \varepsilon_\theta=\varepsilon_\theta^e+\varepsilon_\theta^p. \tag{2}$$

Using Hook's Law and compatibility equations, we get

$$\begin{aligned}\sigma_r&=\frac{E}{1-v^2}\left[\left(\frac{du}{dr}-\varepsilon_r^p\right)+v\left(\frac{u}{r}-\varepsilon_\theta^p\right)\right],\\ \sigma_\theta&=\frac{E}{1-v^2}\left[\left(\frac{u}{r}-\varepsilon_\theta\right)+v\left(\frac{du}{dr}-\varepsilon_r^p\right)\right].\end{aligned} \tag{3}$$

The stresses in both elastic and inelastic regions are governed by the equation of equilibrium,

$$\frac{d\sigma_r}{dr}+\frac{\sigma_r-\sigma_\theta}{r}=0. \tag{4}$$

Substituting Eqs. (3) into Eq. (4), we obtain

$$\frac{d^2u}{dr^2}+\frac{1}{r}\frac{du}{dr}-\frac{1}{r^2}u=F(\varepsilon_r^p;\varepsilon_\theta^p), \tag{5}$$

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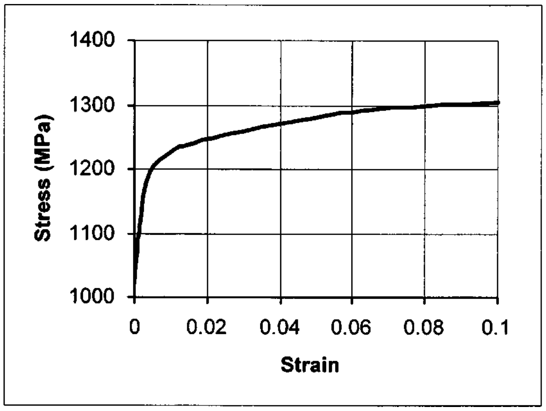

Fig. 1 Universal stress strain curve

$$F=\frac{d\varepsilon_r^p}{dr}+v\,\frac{d\varepsilon_\theta^p}{dr}+\frac{1-v}{r}\,\varepsilon_r^p-\frac{1-v}{r}\,\varepsilon_\theta^p.$$

This is a differential equation for displacement, which can be solved when the function F of the plastic strains ε_θ^p and ε_r^p is known. Plastic strains can be found using the Prandtl-Reuss theory [11], which assumes that

$$d\varepsilon_{ij}^p=d\lambda S_{ij}, \tag{6}$$

where $d\lambda$ is a scalar multiplier (not a constant). This scalar is found using the assumption of a universal stress-strain curve relating to two scalar quantities: the effective stress $\bar{\sigma}$ and the integral of the effective plastic-strain increment $\overline{d\varepsilon^p}$, namely

$$\bar{\sigma}=H\left[\int\overline{d\varepsilon^p}\right], \tag{7}$$

where H is the plastic portion of a uniaxial tensile test curve, defined as a universal stress-strain curve (Fig. 1).

The multiplier $d\lambda$ can be determined as follows:

$$d\varepsilon_{ij}^p d\varepsilon_{ij}^p=(d\lambda)^2 S_{ij}\cdot S_{ij},$$

$$\frac{3}{2}(\overline{d\varepsilon^p})^2=(d\lambda)^2\frac{2}{3}\bar{\sigma}^2.$$

Thus

$$d\lambda=\frac{3}{2}\,\frac{\overline{d\varepsilon^p}}{\bar{\sigma}}=\frac{3}{2}\,\frac{d\bar{\sigma}}{\bar{\sigma}H'} \tag{8}$$

where $H'=d\bar{\sigma}/\overline{d\varepsilon^p}$ is the slope of the universal curve of Eq. (7) at the current value of $\bar{\sigma}$.

The plastic strain increment is given by

$$d\varepsilon_{ij}^p=\frac{3}{2}\,\frac{\overline{d\varepsilon^p}}{\bar{\sigma}}\,S_{ij}. \tag{9}$$

This equation is essential for solving the autofrettage problem. Knowing or assuming $\overline{d\varepsilon^p}$ we can easily calculate the plastic strains $(\varepsilon_r^p;\varepsilon_\theta^p)$ and then use them in F for solving the differential equation (5).

The differential equation (5) can be solved for $u(r)$ by the finite difference method [12]. To this end, let us divide the cylinder wall into N rings at a distance Δr apart. Substituting the central difference form of the derivatives into Eqs. (5), one obtains

$$\frac{1}{(\Delta r)^2}(u_{n+1}-2u_n+u_{n-1})+\frac{1}{2\cdot r_n(\Delta r)}(u_{n+1}-u_{n-1})+\frac{1}{r_n^2}u_n$$
$$=\left(\frac{d\varepsilon_r^p}{dr}+v\,\frac{d\varepsilon_\theta^p}{dr}+\frac{1-v}{r}\,\varepsilon_r^p-\frac{1-v}{r}\,\varepsilon_\theta^p\right)_n$$

or

$$a_n u_{n-1}+b_n u_n+c_n u_{n+1}=S_n. \tag{10}$$

For plane stress, the boundary conditions are

$$(\sigma_r)_{r=b}=0 \quad (\sigma_r)_{r=a}=-P. \tag{11}$$

Using these boundary conditions the stress equations (3) can be written as

$$\left(\frac{du}{dr}\right)_a-(\varepsilon_r^p)_a+v\left(\frac{u_a}{a}-(\varepsilon_\theta^p)_a\right)=-\frac{1-v^2}{E}P, \tag{12}$$

$$\left(\frac{du}{dr}\right)_b-(\varepsilon_r^p)_b+v\left(\frac{u_b}{b}-(\varepsilon_\theta^p)_b\right)=0. \tag{13}$$

Substituting the central difference form of the boundary derivatives into Eqs. (12) and (13), one obtains

$$u_0=u_2+\frac{2(\Delta r)v}{a}u_1-ZK,$$
$$ZK=2(\Delta r)\left((\varepsilon_r^p)_1+v(\varepsilon_\theta^p)_1-\frac{1-v^2}{E}P\right), \tag{14}$$

and

$$u_{N+2}=u_N-\frac{2(\Delta r)v}{b}u_{N+1}+ZKN,$$
$$ZKN=2(\Delta r)((\varepsilon_r^p)_{N+1}+v(\varepsilon_\theta^p)_{N+1}). \tag{15}$$

Substituting Eqs. (14) into Eq. (10), one obtains

$$\left(b_1+a_1\frac{2(\Delta r)v}{a}\right)u_1+(c_1+a_1)u_2=S_1+a_1(ZK)$$

or

$$(b_1)_f u_1+(c_1)_f u_2=(S_1)_f. \tag{16}$$

Substituting Eqs. (15) into Eq. (10), one obtains

$$(a_{N+1}+c_{N+1})u_N+\left(b_{N+1}-c_{N+1}\frac{2(\Delta r)v}{b}\right)u_{N+1}$$
$$=S_{N+1}-c_{N+1}(ZKN)$$

or

$$(a_{N+1})_f u_N+(b_{N+1})_f u_{N+1}=(S_{N+1})_f. \tag{17}$$

Equations (10), (16), and (17) can be written in matrical form

$$A\cdot U=S \tag{18}$$

while

$$U=\begin{bmatrix}u_1\\u_2\\\vdots\\u_N\\u_{N+1}\end{bmatrix}\quad S=\begin{bmatrix}(S_1)_f\\S_2\\\vdots\\S_N\\(S_{N+1})_f\end{bmatrix},$$

$$A=\begin{bmatrix} (b_1)_f & (c_1)_f & & & & & \\ a_2 & b_2 & c_2 & & & & \\ & \cdot & \cdot & \cdot & & & \\ & & \cdot & \cdot & \cdot & & \\ & & & \cdot & \cdot & \cdot & \\ & & & & a_N & b_N & c_N \\ & & & & & (a_{N+1})_f & (b_{N+1})_f \end{bmatrix}.$$

The solution of Eq. (18) determines the displacement distribution throughout the cylinder wall. Knowing the displacement and using Eqs. (3), we can find the stress distribution throughout the cylinder wall.

The numerical solution of autofrettage problems comprises of four stages:

i. pressure loading—plastic strains determination;
ii. pressure unloading—residual stresses determination;
iii. material removal—final residual stresses determination;
iv. elastic strength pressure (ESP) determination.

To start the incremental pressure loading, the critical pressure of plastic yielding is determined from

$$P_{cr}=\frac{\sigma_y(1-(a/b)^2)}{\sqrt{3+(a/b)^4}}. \tag{19}$$

At every incremental step, a convergence process is performed such that

$$f(\sigma_{ij})_{n+1}=\bar{\sigma}_{n+1}\equiv H(\bar{\varepsilon}^p_{n+1})$$

is satisfied, where $f(\sigma_{ij})$ is the function of the von Mises yield criterion.

The convergence process starts with the assumption that $(d\varepsilon^p_{ij})_n=(d\varepsilon^p_{ij})_{n+1}$ and, after finding the new plastic strains, the stresses $(\sigma_{ij})^I_{n+1}$ are determined. Then $\overline{d\varepsilon^p}$ is increased by 10%, the strains $(d\varepsilon^p_{ij})^{II}_{n+1}$ are calculated, and again, the stresses $(\sigma_{ij})^{II}_{n+1}$ are found.

Knowing the variance in effective stresses caused by the $\overline{d\varepsilon^p}$ modification, KC, the "coefficients of influence," is calculated,

$$\begin{aligned} WI&=f(\sigma_{ij})^I-f(\sigma_{ij})^{II}, \\ WJ&=(\bar{\sigma})^I-(\bar{\sigma})^{II}, \\ WM&=f(\sigma_{ij})^{II}-f(\bar{\sigma})^{II}, \\ WG&=\frac{WM}{1+WI/WJ}, \\ KC&=WG/WJ. \end{aligned} \tag{20}$$

The final effective strain increment will be

$$(\overline{d\varepsilon^p})^{II}=(\overline{d\varepsilon^p})^I(1-0.1(KC)) \tag{21}$$

and the final value of the effective strain will be

$$(\bar{\varepsilon}^p)_{n+1}=(\bar{\varepsilon}^p)_n+(\overline{d\varepsilon^p})^{II}. \tag{22}$$

Using those effective strains we obtain the convergence stresses such that

$$f(\sigma_{ij})^{III}_{n+1}=H(\bar{\varepsilon}^p)^{III}_{n+1}.$$

In the unloading stage, pressure is decreased incrementally, while yielding is determined at each step. The yield point in compression depends on the amount of plastic deformation, which occurs during tension (Fig. 2). The curve of Fig. 2 is obtained by a uniaxial tension compression cycling test, during which the stress-strain curve beyond the yielding point is also established (Fig. 3). Figure 3 was generated from the average of the stress-strain curves in the compressive part, starting from the saturation yield point. Figure 3 is used in the following manner: once the yield point is determined from Fig. 2, it is located on the curve of Fig. 3, after which the strain hardening is determined, following the right portion of the curve. This stress-strain curve is the corresponding H function in compression.

When yielding during unloading takes place, the stresses are calculated in the same way as during the pressure loading stage described above. When the pressure reaches zero, we are left only with the residual stresses distribution in the autofrettaged cylinder (Fig. 4).

In order to calculate the residual stresses after machining, an analytic simulation method for changing the cylinder configuration was developed. Using the method of finite differences, the

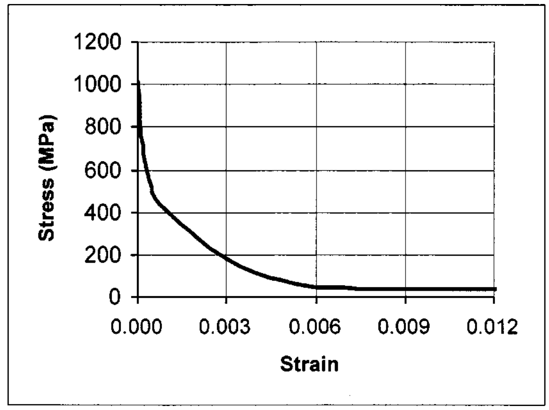

Fig. 2 Compression yield point versus tension plastic deformation

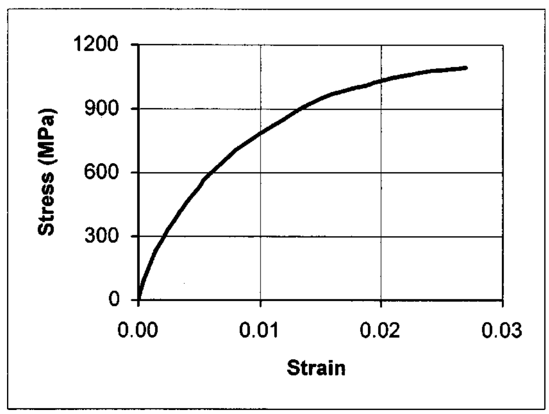

Fig. 3 Compression stress-strain curve beyond the yielding point

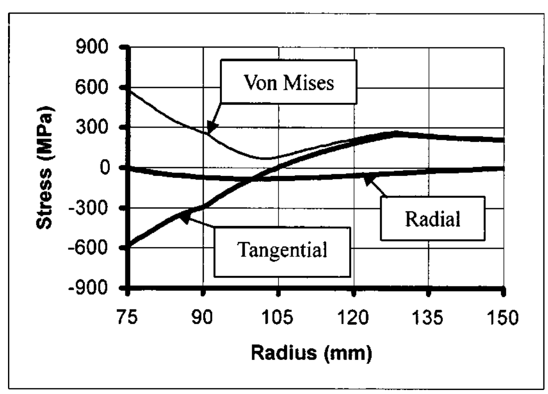

Fig. 4 Residual stresses distribution after pressure unloading

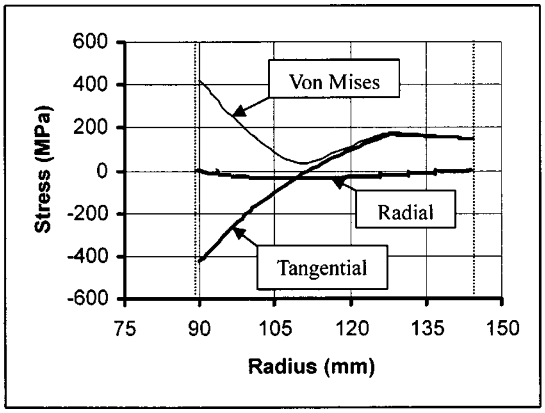

Fig. 5 Residual stresses distribution after machining

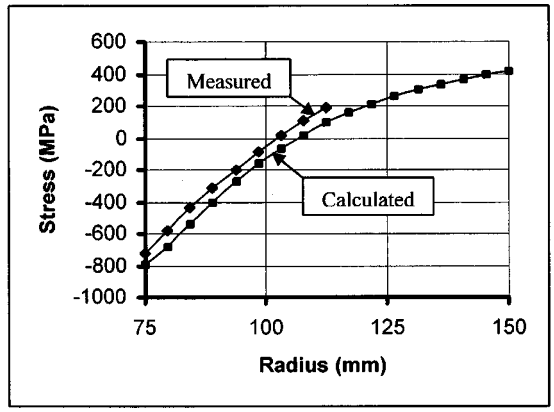

Fig. 7 Measured and calculated tangential stresses

cylinder was divided into N rings, defined by the radii $r_n;n=K,K+1,\ldots,N,N+1$ (at the beginning $K=1$). Machining is simulated by removing rings from the cylinder wall by changing K and N, while the stress distribution is found from using the finite difference method. When changes in K and N bring the cylinder to its desired final shape, the resulting residual stresses distribution (Fig. 5) is obtained.

It should be mentioned that Parker et al. [7] determined the residual stresses after the removal of the material from the external and internal surfaces, and they detected a dependence on the sequence of material removal. Our calculations indicate that such dependence is not obtained from the present method of analysis.

Finally, by increasing the pressure until plastic yielding is achieved, and taking into account the secondary Bauschinger effect (the effect of yielding in compression on the subsequent yield point in tension), we get the maximal stresses distribution (Fig. 6) while the von Mises stress on the bore surface is the maximum allowable internal pressure.

Experimental Results and Discussion

The residual stresses in the autofrettaged cylinder were determined by machining the internal diameter and measuring the strain changes on the outer diameter (the Sachs method) [13]. Using the method developed in this research, the residual stresses in the same cylinder were calculated and a very good correlation of shape and magnitude was found (Fig. 7). The small deviation in the stress values is apparently caused by the longitudinal residual stresses due to swage autofrettage, as was reported by Davidson, Kendal, and Reiner [3]. It should be mentioned that in the present case the initial residual stresses in the cylinder under investigation were in the elastic domain. Consequently, the significant difference between the measured residual stresses by the Sachs method [7] and the calculated one cannot be detected. Nevertheless the present method was employed for comparing the calculated residual stresses with the measured ones by the Sachs method [3] in the same manner as in Ref. [7], and again good correlation was found, as shown in Fig. 8.

Residual stresses in a fully overstrained cylinder in which the ratio of the outer to the inner diameters is 2 (made of SAE 4340 steel the yield strength of which is 1050 MPa) were calculated using the conventional analytical method [3] and the present numerical one, see Fig. 9. It is clearly seen that there is no significant difference between the radial stresses in the two cases, but the present numerically calculated tangential stresses differ considerably, exhibiting 22% lower values on the bore surface. The same phenomenon was reported by Clark [5].

To investigate which parameter affects this difference mostly, the Bauschinger effect was eliminated from the computer program and the resulting stress distribution was calculated, see Fig. 10. In this case there is no significant difference between the two methods and the tangential stresses match completely the analytical residual stresses for the Tresca and von Mises criteria reported by Davidson, Kendall, and Reiner [3]. To investigate how the overstrain level affects the Bauschinger effect, the maximal tangential residual stresses on the bore surface were calculated (Fig. 11) with and without the Bauschinger effect. It is clearly seen that the reduction of the residual stress due to the Bauschinger effect increases with the increase of the overstrain, apparently because of the reversed yielding.

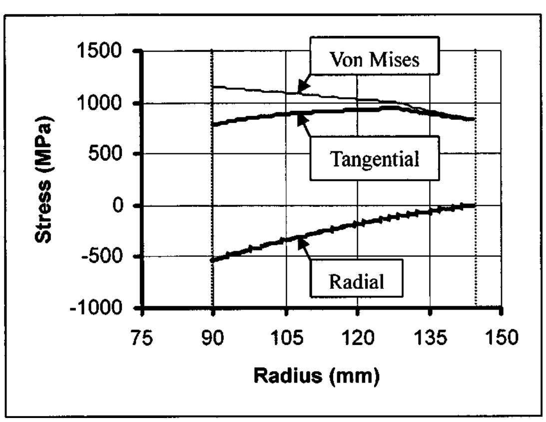

Fig. 6 Pressurized cylinder maximal stresses distribution

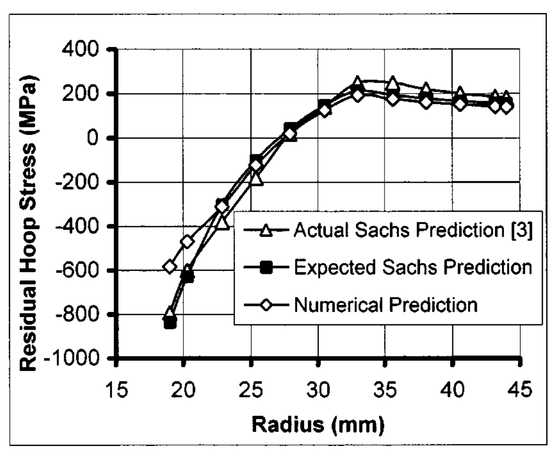

Fig. 8 Overestimate in Sachs' residual stress predictions arising from Bauschinger effect—comparison with experimental results

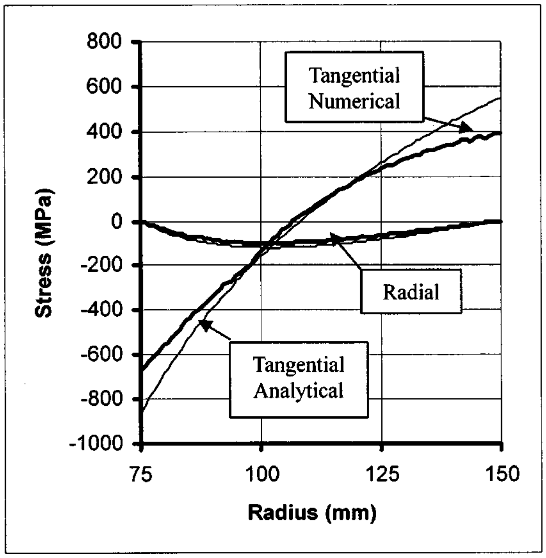

Fig. 9 Analytically and numerically calculated residual stresses distribution

Conclusions

1. Application of the von Mises yield criterion, isotropic strain hardening in conjunction with the Prandtl-Reuss theory, pressure release taking into account the Bauschinger effect, and plane stress conditions were found to be suitable for predicting stress distribution in a thick-walled autofrettaged cylinder.

2. The von Mises yield criterion and strain hardening does not affect the residual stresses distribution significantly.

3. The Bauschinger effect affects significantly the tangential residual stress distribution, expressly with overstrain increase.

4. The finite difference method is very effective in solving autofrettage problems numerically, especially in numerical simulation of cylinder machining.

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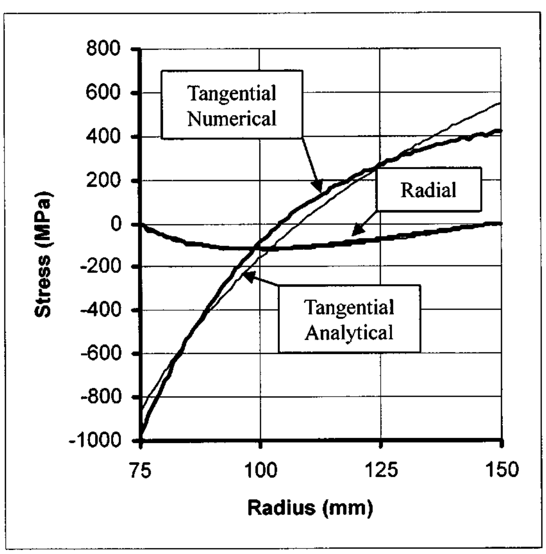

Fig. 10 Residual stresses distribution while Bauschinger effect is eliminatred

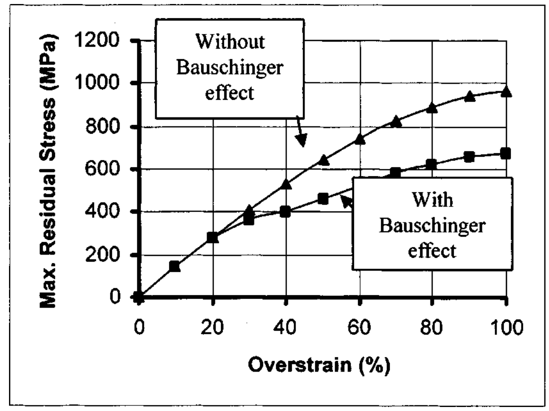

Fig. 11 Maximal tangential residual stresses versus overstrain

Nomenclature

σ = stress
P = pressure
a = bore radius
b = outside radius
r = variable radius
ε = strain
E = Young's modulus
v = Poisson's ratio
u = radial displacement
S = mean stress
$\bar{\sigma}$ = effective stress
$\bar{\varepsilon}$ = effective strain

Subscripts

r = radial directions
θ = tangential directions
p = plastic zone
e = elastic zone

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